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On Interpolation, Summation, and the Adjustment of Numerical Tables. By W. S. B. WOOLHOUSE, F.R.A.S.; F.I.A., &c.

SUMMATION.

THE three heads, into which it has been found expedient to divide the general subject of the present paper, all relate to the consideration of a series of numerical quantities which are supposed to represent, either accurately or approximately, some known or implied law which necessarily has an influence in regulating their progression. Assuming that we only know the values of certain terms of the series of quantities, distributed at stated equidistant intervals, it has been shown that all the intermediate or absent terms are most accurately and expeditiously determined by the process of interpolation, already discussed and practically developed. In some calculations, however, the intermediate values may be considered to be mere auxiliaries, and the ulterior object of the interpolation may simply be that of obtaining the sum of the complete series of quantities so found. Now, the final result of this last summation may be directly determined from the given values and their differences by a special formula, without incurring the labour of interpolating the absent terms of the series. It is this particular formula, and some remarkable and useful modifications of it, which we now propose to investigate.

The general expression (γ) for an interpolated quantity V_x is

$$
\begin{aligned}
V_x = & \{(V)+(x-\tfrac{1}{2})a_1\} + \frac{x(x-1)}{2}\left\{(b)+\frac{x-\frac{1}{2}}{3}c_1\right\} \\
& + \frac{x(x^2-1)(x-2)}{2.3.4}\left\{(d)+\frac{x-\frac{1}{2}}{5}e_1\right\} \\
& + \frac{x(x^2-1)(x^2-4)(x-3)}{2\ldots.6}\left\{(f)+\frac{x-\frac{1}{2}}{7}g_1\right\} \\
& + \frac{x(x^2-1)(x^2-4)(x^2-9)(x-4)}{2\ldots.8}\left\{(h)+\frac{x-\frac{1}{2}}{9}i\right\} \text{ \&c.}
\end{aligned}
$$

On examination, it will here be found that the several coefficients have like values when x becomes $1-x$; but those of a_1, c_1, e_1, g_1, i_1, will appear with reverse signs. Thus, for example, the coefficient of g_1 may be written

$$\frac{(x-3)(x-2)(x-1)x(x+1)(x+2)}{2\ldots.6}\cdot\frac{x-\frac{1}{2}}{7},$$

and the several factors obviously recur with a contrary algebraic sign when x is replaced by $1-x$. It is, therefore, evident that the result of either a summation or an integral, from $x=0$ to $x=1$, will vanish with respect to a_1, c_1, e_1, g_1, i_1, the coefficients of which contain an odd number of these changing factors.

Let m be the number of subdivisions of the interpolation, so that $m-1$ values of V are interposed between V_0 and V_1, and the number of values will be $m+1$ if V_0 and V_1 be included. The corresponding $m+1$ values of x are

$$0,\ \frac{1}{m},\ \frac{2}{m},\ \frac{3}{m},\ \ldots.\ \frac{m}{m}.$$

To proceed with the summation, it will first be requisite to determine expressions for the sums of the powers of the numbers 1, 2, 3, ... m. Let the prefix Σ denote a summation of the values of any quantity for all integral values of m from 1 to m, or from m down to 1. Then, in accordance with this definition,

$$
\begin{aligned}
\Sigma\{m^{n+1}-(m-1)^{n+1}\} = & \ m^{n+1}-(m-1)^{n+1} \\
& +(m-1)^{n+1}-(m-2)^{n+1} \\
& +(m-2)^{n+1}-\text{\&c.} \\
& \qquad \text{\&c.} \qquad \text{\&c.} \\
& +2^{n+1}-1 \\
& +1-0 \\
= & \ m^{n+1}.
\end{aligned}
$$

In place of the quantity under Σ, substitute the equivalent expansion

$$(n+1)m^n - \frac{(n+1)n}{2}m^{n-1} + \frac{(n+1)n(n-1)}{2.3}m^{n-2} - \text{\&c.},$$

and divide by $n+1$, then

$$\Sigma m^n - \frac{n}{2}\Sigma m^{n-1} + \frac{n(n-1)}{2.3}\Sigma m^{n-2} - \&c. = \frac{m^{n+1}}{n+1}$$

$$\therefore\ \Sigma m^n = \frac{m^{n+1}}{n+1} + \frac{n}{2}\Sigma m^{n-1} - \frac{n(n-1)}{2.3}\Sigma m^{n-2}$$

$$+ \frac{n(n-1)(n-2)}{2.3.4}\Sigma m^{n-3} - \&c.$$

where the factors after the first term are the binomial coefficients of the order *n*, divided respectively by 2, 3, 4, &c.

Hence, putting 1, 2, 3, &c., successively for *n*, we obtain

$$^{*}\Sigma m = \frac{m^2+m}{2}$$

$$\Sigma m^2 = \frac{2m^3+3m^2+m}{6}$$

$$\Sigma m^3 = \frac{m^4+2m^3+m^2}{4}$$

$$\Sigma m^4 = \frac{6m^5+15m^4+10m^3-m}{30}$$

$$\Sigma m^5 = \frac{2m^6+6m^5+5m^4-m^2}{12}$$

$$\Sigma m^6 = \frac{6m^7+21m^6+21m^5-7m^3+m}{42}$$

$$\Sigma m^7 = \frac{3m^8+12m^7+14m^6-7m^4+2m^2}{24}$$

$$\Sigma m^8 = \frac{10m^9+45m^8+60m^7-42m^5+20m^3-3m}{90}.$$

---

* If $h=2m+1$; $k=m(m+1)$; then

$$\Sigma m = \frac{k}{2}$$

$$\Sigma m^2 = \frac{hk}{6}$$

$$\Sigma m^3 = \frac{k^2}{4}$$

$$\Sigma m^4 = \frac{hk}{30}(3k-1)$$

$$\Sigma m^5 = \frac{k^2}{12}(2k-1)$$

$$\Sigma m^6 = \frac{hk}{42}(3k^2-3k+1)$$

$$\Sigma m^7 = \frac{k^2}{24}(3k^2-4k+2)$$

$$\Sigma m^8 = \frac{hk}{90}(5k^3-10k^2+9k-3)$$

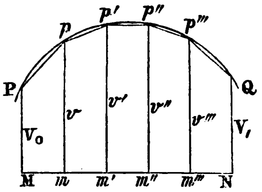
$$\Sigma m^9 = \frac{k^2}{20}(2k^3-5k^2+6k-3).$$

The following properties are readily inferred :—

$$(\Sigma m)^2 = \Sigma m^3$$

$$(\Sigma m^3)^2 = \tfrac{1}{2}(\Sigma m^5 + \Sigma m^7).$$

Now, for the sake of symmetry and the facility of carrying on the summation into the adjoining intervals, it will be desirable to reject one-half of the extreme values V_0, V_1, for which $x=0$, 1, respectively.



For example, suppose that $m=5$, and in the diagram let the ordinates PM, QN, represent geometrically the values of V_0, V_1, and pm, $p'm'$, $p''m''$, $p'''m'''$ the interpolated values denoted by v, v', v'', v'''. Then the summation under the proposed modification will be

$$\Sigma(V')_{0\ldots.1}=\tfrac{1}{2}V_0+v+v'+v''+v'''+\tfrac{1}{2}V_1$$

$$=\frac{V_0+v}{2}+\frac{v+v'}{2}+\frac{v'+v''}{2}+\frac{v''+v'''}{2}+\frac{v'''+V_1}{2},$$

each term of which now expresses a mean value (V') of two consecutive ordinates.

Let the extremities of the ordinates be joined consecutively, so as to form the polygonal line $P\,p\,p'\,p''\,p'''\,Q$, and let the trapezoidal areas contained between the successive ordinates be denoted by a_1, a_2, a_3, a_4, a_5. Then, assuming that $Mm=mm'=\&c.=1$, we shall have

$$\frac{V_0+v}{2}=a_1,\quad \frac{v+v'}{2}=a_2,\quad \frac{v'+v''}{2}=a_3,\ \&c.;$$

and hence

$$\Sigma(V')_{0\ldots.1}=a_1+a_2+a_3+a_4+a_5=P'.$$

Thus, supposing the distance between the interpolated ordinates to be the linear unit, we conclude generally that when, as proposed, one-half of the extreme terms are rejected from the summation, the result will express, geometrically, the polygonal area contained between V_0 and V_1.

By modifying accordingly the signification of $\Sigma(x^n)$, that is, by including only one-half of the extreme terms in the summation, and distinguishing the same by means of a parenthesis, observing that the values of x involved in the summation are

$$0,\ \frac{1}{m},\ \frac{2}{m},\ \frac{3}{m},\ \ldots.\frac{m}{m},$$

we shall then obviously have

$$\Sigma(x^n)=\frac{\Sigma m^n}{m^n}-\tfrac{1}{2}(0^n+1^n)=\frac{\Sigma m^n}{m^n}-\tfrac{1}{2};$$

and thus, from the preceding expressions (4), we deduce

$$\begin{aligned}
&\Sigma(1)=m\\
&\Sigma(x)=\frac{m}{2}\\
&\Sigma(x^2)=\frac{m}{3}+\frac{1}{6m}\\
&\Sigma(x^3)=\frac{m}{4}+\frac{1}{4m}\\
&\Sigma(x^4)=\frac{m}{5}+\frac{1}{3m}-\frac{1}{30m^3}\\
&\Sigma(x^5)=\frac{m}{6}+\frac{5}{12m}-\frac{1}{12m^3}\\
&\Sigma(x^6)=\frac{m}{7}+\frac{1}{2m}-\frac{1}{6m^3}+\frac{1}{42m^5}\\
&\Sigma(x^7)=\frac{m}{8}+\frac{7}{12m}-\frac{7}{24m^3}+\frac{1}{12m^5}\\
&\Sigma(x^8)=\frac{m}{9}+\frac{2}{3m}-\frac{7}{15m^3}+\frac{2}{9m^5}-\frac{1}{30m^7}.
\end{aligned}$$

We now proceed to apply these to the foregoing formula for V_x, in which it has already been explained that the result will be evanescent as regards a_1, c_1, e_1, &c.

The other terms, when expanded, are

$$\begin{aligned}
(\mathrm{V})&+\frac{(b)}{2}(x^2-x)\\
&+\frac{(d)}{2.3.4}(x^4-2x^3-x^2+2x)\\
&+\frac{(f)}{2\ldots.6}(x^6-3x^5-5x^4+15x^3+4x^2-12x)\\
&+\frac{(h)}{2\ldots.8}(x^8-4x^7-14x^6+56x^5+49x^4-196x^3-36x^2+144x).
\end{aligned}$$

The severally corresponding coefficients for the required summation are therefore deduced as follows:—

Coefficient of $(\mathrm{V})=m$.

$$\begin{aligned}
\text{Ditto of }\frac{(b)}{2}=\ &\frac{m}{3}+\frac{1}{6m}\\
&-\frac{m}{2}\\
\hline
=-&\frac{m}{6}+\frac{1}{6m}=-\frac{m^2-1}{6m}.
\end{aligned}$$

$$
\begin{aligned}
\text{Ditto of } \frac{(d)}{2.3.4} = \; & \frac{m}{5} + \frac{1}{3m} - \frac{1}{30m^3} \\
& -\frac{m}{2} - \frac{1}{2m} \\
& -\frac{m}{3} - \frac{1}{6m} \\
& +m \\
= \; & \frac{11m}{30} - \frac{1}{3m} - \frac{1}{30m^3} = \frac{11m^4-10m^2-1}{30m^3} \\
& = \frac{(m^2-1)(11m^2+1)}{30m^3}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Ditto of } \frac{(f)}{2\ldots.6} = \; & \frac{m}{7} + \frac{1}{2m} - \frac{1}{6m^3} + \frac{1}{42m^5} \\
& -\frac{m}{2} - \frac{5}{4m} + \frac{1}{4m^3} \\
& -m - \frac{5}{3m} + \frac{1}{6m^3} \\
& +\frac{15m}{4} + \frac{15}{4m} \\
& +\frac{4m}{3} + \frac{2}{3m} \\
& -6m \\
= \; & -\frac{191m}{84} + \frac{2}{m} + \frac{1}{4m^3} + \frac{1}{42m^5} = -\frac{191m^6-168m^4-21m^2-2}{84m^5} \\
& = -\frac{(m^2-1)(191m^4+23m^2+2)}{84m^5}
\end{aligned}
$$

$$
\begin{aligned}
\text{Ditto of } \frac{(h)}{2\ldots.8} = \; & \frac{m}{9} + \frac{2}{3m} - \frac{7}{15m^3} + \frac{2}{9m^5} - \frac{1}{30m^7} \\
& -\frac{m}{2} - \frac{7}{3m} + \frac{7}{6m^3} - \frac{1}{3m^5} \\
& -2m - \frac{7}{m} + \frac{7}{3m^3} - \frac{1}{3m^5} \\
& +\frac{28m}{3} + \frac{70}{3m} - \frac{14}{3m^3} \\
& +\frac{49m}{5} + \frac{49}{3m} - \frac{49}{30m^3} \\
& -49m - \frac{49}{m} \\
& -12m - \frac{6}{m}
\end{aligned}
$$

$$\frac{\begin{array}{l}+72m\end{array}}{}$$

$$=\frac{2497m}{90}-\frac{24}{m}-\frac{98}{30m^3}-\frac{4}{9m^5}-\frac{1}{30m^7}$$

$$=\frac{2497m^8-2160m^6-294m^4-40m^2-3}{90m^7}$$

$$=\frac{(m^2-1)(2497m^6+337m^4+43m^2+3)}{90m^7}.$$

Hence the formula which expresses the sum of the $m-1$ interpolated values, together with one-half of the two extreme values V_0, V_1, is

$$\Sigma(V')_{0\ldots\ldots 1}=m(V)-\frac{m^2-1}{12m}(b)+\frac{(m^2-1)(11m^2+1)}{720m^3}(d)$$

$$-\frac{(m^2-1)(191m^4+23m^2+2)}{60480m^5}(f)$$

$$+\frac{(m^2-1)(2497m^6+337m^4+43m^2+3)}{3628800m^7}(h)\ \ldots(a).$$

Dividing by m we get the average of m values $=$

$$(V)-\frac{m^2-1}{12m^2}(b)+\frac{(m^2-1)(11m^2+1)}{720m^4}(d)$$

$$-\frac{(m^2-1)(191m^4+23m^2+2)}{60480m^6}(f)$$

$$+\frac{(m^2-1)(2497m^6+337m^4+43m^2+3)}{3628800m^8}(h).$$

When m is supposed to be indefinitely great, this becomes

$$(V)-\frac{1}{12}(b)+\frac{11}{720}(d)-\frac{191}{60480}(f)+\frac{2497}{3628800}(h)\ \ldots\ (b)$$

which is the same result as that determined by the integral calculus, and expresses the sum of an infinite number of ordinate spaces, or the curvilinear area contained between the ordinates V_0 and V_1, when the distance between them is unity.

The formula (a) may be applied so as to include any number $n+1$ of primitive values $V_0, V_1, V_2, \ldots\ldots V_n$. Denoting the respective coefficients by m, $-B$, D, $-F$, H, for the sake of brevity, the expression is

$$\Sigma(V')_{0\ldots\ldots 1}=m(V)-B(b)+D(d)-F(f)+H(h);$$

and when extended, so as to comprise all the n intervals from V_0 to V_n, it gives

$$\Sigma(V')_{0\ldots\ldots n}=m\Sigma(V)-B\Sigma(b)+D\Sigma(d)-F\Sigma(f)+H\Sigma(h)$$

$$=m\Sigma(V)-B\{(a_n)-(a_0)\}+D\{c_n)-(c_0)\}$$

$$-F\{(e_n)-(e_0)\}+H\{(g_n)-(g_0)\}\ \ldots\ (5).$$

To employ this general formula, the values of (a_0), (a_n), (b_0), (b_n), &c., as ordinarily derived by a direct process of differencing, require a previous knowledge or calculation of certain values of V which precede V_0 and follow V_n. This, however, may be practically dispensed with by supposing the differences to be extended upwards and downwards, so that the last order of differences shall retain the same constant value, or that the next following order of differences, and, of course, all succeeding orders, shall vanish or be disregarded. Thus, suppose the quantities V_0, V_1, V_2, ... V_n, to be differenced with a notation as follows:—

$$
\begin{array}{|l|l|l|l|l|l|l|l|l}
V_0 & & & & & & & & \\
 & a & & & & & & & \\
V_1 & & b & & & & & & \\
 & a_1 & & c & & & & & \\
V_2 & & b_1 & & d & & & & \\
 & a_2 & & c_1 & & e & & & \\
V_3 & & b_2 & & d_1 & & f & & \\
 & a_3 & & c_2 & & e_1 & & g & \\
V_4 & & b_3 & & d_2 & & f_1 & & h \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
V_{n-4} & & b'_{-3} & & d'_{-2} & & f'_{-1} & & h' \\
 & a'_{-3} & & c'_{-2} & & e'_{-1} & & g' & \\
V_{n-3} & & b'_{-2} & & d'_{-1} & & f' & & \\
 & a'_{-2} & & c'_{-1} & & e' & & & \\
V_{n-2} & & b'_{-1} & & d' & & & & \\
 & a'_{-1} & & c' & & & & & \\
V_{n-1} & & b' & & & & & & \\
 & a' & & & & & & & \\
V_n & & & & & & & &
\end{array}
$$

Then we shall have

$$
\begin{aligned}
a_{n-1} &= a' \\
a_{n+1} &= a' + b' + c' + d' + e' + f' + g' + h' \\
\therefore (a_n) &= a' + \tfrac{1}{2}b' + \tfrac{1}{2}c' + \tfrac{1}{2}d' + \tfrac{1}{2}e' + \tfrac{1}{2}f' + \tfrac{1}{2}g' + \tfrac{1}{2}h'
\end{aligned}
$$

$$
\begin{aligned}
a_{-1} &= a - b + c - d + e - f + g - h \\
a_{+1} &= a \\
\therefore (a_0) &= a - \tfrac{1}{2}b + \tfrac{1}{2}c - \tfrac{1}{2}d + \tfrac{1}{2}e - \tfrac{1}{2}f + \tfrac{1}{2}g - \tfrac{1}{2}h
\end{aligned}
$$

$$
\begin{aligned}
c_{n-1} &= c' + d' + e' + f' + g' + h' \\
c_{n+1} &= c' + 2d' + 3e' + 4f' + 5g' + 6h' \\
\therefore (c_n) &= c' + \tfrac{3}{2}d' + 2e' + \tfrac{5}{2}f' + 3g' + \tfrac{7}{2}h'
\end{aligned}
$$

$$
\begin{aligned}
c_{-1} &= c - 2d + 3e - 4f + 5g - 6h \\
c_{+1} &= c - d + e - f + g - h \\
\therefore (c_0) &= c - \tfrac{3}{2}d + 2e - \tfrac{5}{2}f + 3g - \tfrac{7}{2}h
\end{aligned}
$$

$$
\begin{array}{l|l}
e_{n-1} = e' + 2f' + 3g' + 4h' & g_{n-1} = g' + 3h' \\
e_{n+1} = e' + 3f' + 6g' + 10h' & g_{n+1} = g' + 4h' \\
\therefore (e_n) = e' + \tfrac{5}{2}f' + \tfrac{9}{2}g' + 7h' & (g_n) = g' + \tfrac{7}{2}h' \\
 & \\
e_{-1} = e - 3f + 6g - 10h & g_{-1} = g - 4h \\
e_{+1} = e - 2f + 3g - 4h & g_{+1} = g - 3h \\
\therefore (e_0) = e - \tfrac{5}{2}f + \tfrac{9}{2}g - 7h & (g_0) = g - \tfrac{7}{2}h.
\end{array}
$$

And since $\Sigma(b)=(a_n)-(a_0)$, $\Sigma(d)=(c_n)-(c_0)$, &c., these values give

$$\begin{aligned}\Sigma(b)&=(a'-a)+\tfrac{1}{2}(b'+b)+\tfrac{1}{2}(c'-c)+\tfrac{1}{2}(d'+d)+\tfrac{1}{2}(e'-e)\\&\quad+\tfrac{1}{2}(f'+f)+\tfrac{1}{2}(g'-g)+\tfrac{1}{2}(h'+h)\\ \Sigma(d)&=(c'-c)+\tfrac{3}{2}(d'+d)+2(e'-e)+\tfrac{5}{2}(f'+f)+3(g'-g)+\tfrac{7}{2}(h'+h)\\ \Sigma(f)&=(e'-e)+\tfrac{5}{2}(f'+f)+\tfrac{9}{2}(g'-g)+7(h'+h)\\ \Sigma(h)&=(g'-g)+\tfrac{7}{2}(h'+h).\end{aligned}$$

The foregoing expression (5) for $\Sigma(V')_{0\ldots n}$ hence becomes=

$$\begin{aligned}&m\Sigma(V)-B(a'-a)-\tfrac{1}{2}B(b'+b)-(\tfrac{1}{2}B-D)(c'-c)\\&-(\tfrac{1}{2}B-\tfrac{3}{2}D)(d'+d)-(\tfrac{1}{2}B-2D+F)(e'-e)\\&-(\tfrac{1}{2}B-\tfrac{5}{2}D+\tfrac{5}{2}F)(f'+f)-(\tfrac{1}{2}B-3D+\tfrac{9}{2}F-H)(g'-g)\\&-(\tfrac{1}{2}B-\tfrac{7}{2}D+7F-\tfrac{7}{2}H)(h'+h);\end{aligned}$$

that is, after substituting the values of **B, D, F, H**, which are the coefficients of the formula (*a*),

$$\begin{aligned}\Sigma(V')_{0\ldots\ldots n}&=m\{(V_0+V_1\ldots\ldots+V_n)-\tfrac{1}{2}(V_n+V_0)\}-\frac{m^2-1}{12m}(a'-a)\\&-\frac{m^2-1}{24m}(b'+b)-\frac{(m^2-1)(19m^2-1)}{720m^3}(c'-c)-\frac{(m^2-1)(9m^2-1)}{480m^3}(d'+d)\\&-\frac{(m^2-1)(863m^4-145m^2+2)}{60480m^5}(e'-e)-\frac{(m^2-1)(275m^4-61m^2+2)}{24192m^5}(f'+f)\\&-\frac{(m^2-1)(33953m^6-9247m^4+497m^2-3)}{3628800m^7}(g'-g)\\&-\frac{(m^2-1)(8183m^6-2617m^4+197m^2-3)}{1036800m^7}(h'+h)\ \ldots\ (A).^*\end{aligned}$$

By dividing by *mn* we get the mean value of V; and when *m* is made ∞, the result multiplied by *n* gives the curvilinear area bounded by V_0 and V_n=

$$\begin{aligned}&(V_0+V_1\ldots\ldots+V_n)-\tfrac{1}{2}(V_n+V_0)-\frac{1}{12}(a'-a)-\frac{1}{24}(b'+b)\\&-\frac{19}{720}(c'-c)-\frac{3}{160}(d'+d)-\frac{863}{60480}(e'-e)-\frac{275}{24192}(f'+f)\\&-\frac{33953}{3628800}(g'-g)-\frac{8183}{1036800}(h'+h)\ \ldots\ldots\ (B)\end{aligned}$$

which is the formula usually employed for calculating quadratures where analytical integrations are impracticable. *See* De Morgan's

---

* This expression includes only one-half of the extreme values V_0, V_n. To obtain $\Sigma V'_{0\ldots\ldots n}$, including all the $mn+1$ values, primitive and interpolated, $\tfrac{1}{2}(V_n+V_0)$ should be added, or the first term of the formula should be replaced by

$$m(V_0+V_1\ \ldots\ldots\ +V_n)-\frac{m-1}{2}(V_n+V_0).$$

Differential and Integral Calculus, p. 313, art. 181, where the series is given as far as $(e'-e)$.

In order that the investigation may be as complete as possible, we have carried it as far as the mean eighth order of differences, which is more ample than can be required, except for the purpose of ascertaining the magnitude of the terms rejected.

For calculations in general, it will not be requisite to go beyond the fourth order, and, perhaps, in no case will it be desirable to proceed further than the sixth order of differences. Thus, dividing (A) by *mn*, we shall have for the average of *mn* values of (V′), in which it has been considered sufficient to include the two leading terms of the several numerators,

$$\frac{\Sigma(\mathrm{V}')_{0\ldots\ldots n}}{mn}=\frac{(\mathrm{V}_0+\mathrm{V}_1\ldots\ldots+\mathrm{V}_n)-\frac{1}{2}(\mathrm{V}_n+\mathrm{V}_0)}{n}$$

$$-\frac{(a'-a)+\frac{1}{2}(b'+b)+\frac{19}{60}(c'-c)+\frac{9}{40}(d'+d)+\frac{863}{5040}(e'-e)+\frac{275}{2016}(f'+f)}{12n}$$

$$+\frac{1}{m^2}\cdot\frac{(a'-a)+\frac{1}{2}(b'+b)+\frac{1}{3}(c'-c)+\frac{1}{4}(d'+d)+\frac{1}{5}(e'-e)+\frac{1}{6}(f'+f)}{12n}\quad\ldots\text{(C)}.$$

The differences *a, b, c*, &c., may be found in terms of the primitive values by the simple algebraic operation of differencing, and the order of the symbols will be preserved by using the common rule of changing signs and adding, as in the following specimen:—

	$\Delta_1=a$	$\Delta_2=b$	$\Delta_3=c$	$\Delta_4=d$	
V_0	$-\mathrm{V}_0+\mathrm{V}_1$	$\mathrm{V}_0-2\mathrm{V}_1+\mathrm{V}_2$	$-\mathrm{V}_0+3\mathrm{V}_1-3\mathrm{V}_2+\mathrm{V}_3$	$\mathrm{V}_0-4\mathrm{V}_1+6\mathrm{V}_2-4\mathrm{V}_3+\mathrm{V}_4$	&c.
V_1	$-\mathrm{V}_1+\mathrm{V}_2$	$\mathrm{V}_1-2\mathrm{V}_2+\mathrm{V}_3$	$-\mathrm{V}_1+3\mathrm{V}_2-3\mathrm{V}_3+\mathrm{V}_4$	$\mathrm{V}_1-4\mathrm{V}_2+6\mathrm{V}_3-4\mathrm{V}_4+\mathrm{V}_5$	
V_2	$-\mathrm{V}_2+\mathrm{V}_3$	$\mathrm{V}_2-2\mathrm{V}_3+\mathrm{V}_4$	$-\mathrm{V}_2+3\mathrm{V}_3-3\mathrm{V}_4+\mathrm{V}_5$	&c.	
V_3	$-\mathrm{V}_3+\mathrm{V}_4$	$\mathrm{V}_3-2\mathrm{V}_4+\mathrm{V}_5$	&c.		
V_4	$-\mathrm{V}_4+\mathrm{V}_5$	&c.			
V_5	&c.				

On examining the process, it will at once be seen that the numeral coefficients, irrespective of sign, are made up according to the following table, in which the numbers in each column are deduced by respectively adding the number which stands in the preceding column to that which is immediately above it, until the said numbers are severally exhausted.

Numerical Development of the Coefficients.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120
		1	4	10	20	35	56	84	120	165	220	286	364	455	560
			1	5	15	35	70	126	210	330	495	715	1001	1365	1820
				1	6	21	56	126	252	462	792	1287	2002	3003	4368
					1	7	28	84	210	462	924	1716	3003	5005	8008
						1	8	36	120	330	792	1716	3432	6435	11440
							1	9	45	165	495	1287	3003	6435	12870
								1	10	55	220	715	2002	5005	11440
									1	11	66	286	1001	3003	8008
										1	12	78	364	1365	4368
											1	13	91	455	1820
												1	14	105	560
													1	15	120
														1	16
															1

These coefficients, which are here extended as far as the sixteenth order of differences for the purpose of reference, are evidently those given by the binomial theorem, since by the operation of successive multiplication, these are obviously formed in precisely the same manner.

Hence, according as n is odd or even, we have

$$\Delta_n = \left\{ \begin{array}{l} -V_0+nV_1-n\cdot\frac{n-1}{2}V_2+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}V_3-\&c. \\ V_0+nV_1+n\cdot\frac{n-1}{2}V_2+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}V_3+\&c. \end{array} \right\} \quad \ldots \quad (6).^*$$

Proceeding as far as the sixth order of differences, the following are therefore the several values:—

$$\begin{aligned} a &= -V_0+V_1 \\ b &= V_0-2V_1+V_2 \\ c &= -V_0+3V_1-3V_2+V_3 \\ d &= V_0-4V_1+6V_2-4V_3+V_4 \\ e &= -V_0+5V_1-10V_2+10V_3-5V_4+V_5 \\ f &= V_0-6V_1+15V_2-20V_3+15V_4-6V_5+V_6 \\ \&c. & \qquad \&c. \end{aligned}$$

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* If for V_0, V_1, V_2, &c., there be substituted their respective numeral coefficients, the formula (6) in either case will express the nth difference of those coefficients, which, being determined by a continuation of the process, must evidently be the central coefficient of the $2n$th order of the table. We thus arrive in a most simple manner at the curious summation

$$1+n^2+\left(n.\frac{n-1}{2}\right)^2+\left(n.\frac{n-1}{2}.\frac{n-2}{3}\right)^2+\&c.$$

$$=\text{central coefficient of } \Delta_{2n}$$

$$=2n.\frac{2n-1}{2}.\frac{2n-2}{3}\ldots\frac{n+1}{n} \quad \ldots\ldots \quad (7).$$

and these may, if required, be further extended by inspection of the preceding table.

For certain special objects, such as the computation of areas, volumes, tonnage of ships, &c., it may be desirable to practically utilize the formula (C) by resolving a few particular cases, so as to express the results wholly in terms of V_0, V_1, V_2, &c.

1. Let there be two intervals, or three given values; then $n=2$, and

$$a=-V_0+V_1, \quad a'=-V_1+V_2, \quad b=b'=V_0-2V_1+V_2$$

$$\begin{aligned} \therefore a'-a &= V_0-2V_1+V_2 \\ b'+b &= 2(V_0-2V_1+V_2) \end{aligned}$$

and the average of $2m$ values $=$

$$\frac{(V_0+V_2)+4V_1}{6} + \frac{(V_0+V_2)-2V_1}{12m^2} \quad \ldots \quad (c_2).$$

2. Let there be three intervals, or four given values; then $n=3$, and

$$\begin{aligned} a &= -V_0+V_1 & b &= V_0-2V_1+V_2 \\ a' &= -V_2+V_3 & b' &= V_1-2V_2+V_3 \end{aligned}$$

$$c=c'=-V_0+3V_1-3V_2+V_3$$

$$\begin{aligned} \therefore a'-a &= V_0-V_1-V_2+V_3 \\ b'+b &= V_0-V_1-V_2+V_3 \\ c'-c &= 0 \end{aligned}$$

and the average of $3m$ values $=$

$$\frac{(V_0+V_3)+3(V_1+V_2)}{8} + \frac{(V_0+V_3)-(V_1+V_2)}{24m^2} \quad \ldots \quad (c_3).$$

3. With four intervals, or five values, $n=4$;

$$\begin{aligned} a'-a &= V_0-V_1 \quad \ast \quad -V_3+V_4 \\ b'+b &= V_0-2V_1+2V_2-2V_3+V_4 \\ c'-c &= V_0-4V_1+6V_2-4V_3+V_4 \\ d'+d &= 2(V_0-4V_1+6V_2-4V_3+V_4) \end{aligned}$$

and the average of $4m$ values $=$

$$\frac{7(V_0+V_4)+32(V_1+V_3)+12V_2}{90} + \frac{7(V_0+V_4)-16(V_1+V_3)+18V_2}{144m^2} \quad \ldots\ldots \quad (c_4).$$

4. With five intervals, or six values, $n=5$;

$$\begin{aligned} a'-a &= V_0-V_1 \quad \ast \quad \ast \quad -V_4+V_5 \\ b'+b &= V_0-2V_1+V_2+V_3-2V_4+V_5 \\ c'-c &= V_0-3V_1+2V_2+2V_3-3V_4+V_5 \\ d'+d &= V_0-3V_1+2V_2+2V_3-3V_4+V_5 \\ e'-e &= 0 \end{aligned}$$

and the mean of $5m$ values$=$

$$\frac{19(V_0+V_5)+75(V_1+V_4)+50(V_2+V_3)}{288}$$

$$+\frac{5(V_0+V_5)-9(V_1+V_4)+4(V_2+V_3)}{144m^2} \quad \ldots\ldots \quad (c_5).$$

5. With six intervals, or seven given values, $n=6$;

$$\begin{aligned}
a'-a &= V_0-V_1 \quad * \quad * \quad * \quad -V_5+V_6\\
b'+b &= V_0-2V_1+V_2 \quad * \quad +V_4-2V_5+V_6\\
c'-c &= V_0-3V_1+3V_2-2V_3+3V_4-3V_5+V_6\\
d'+d &= V_0-4V_1+7V_2-8V_3+7V_4-4V_5+V_6\\
e'-e &= V_0-6V_1+15V_2-20V_3+15V_4-6V_5+V_6\\
f'+f &= 2(V_0-6V_1+15V_2-20V_3+15V_4-6V_5+V_6)
\end{aligned}$$

and the mean of $6m$ values$=$

$$\frac{41(V_0+V_6)+216(V_1+V_5)+27(V_2+V_4)+272V_3}{840}$$

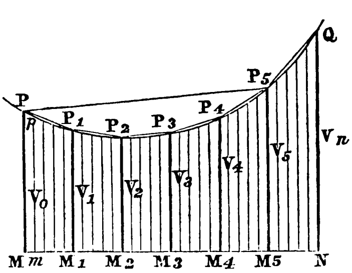
$$+\frac{157(V_0+V_6)-432(V_1+V_5)+675(V_2+V_4)-800V_3}{4320m^2},$$

or (with the small variation $\frac{f}{840}-\frac{37f}{4320m^2}$, for the purpose of simplification)$=$

$$\frac{(V_0+V_6)+5(V_1+V_5)+(V_2+V_4)+6V_3}{20}$$

$$+\frac{4(V_0+V_6)-7(V_1+V_5)+4(V_2+V_4)-2V_3}{144m^2} \quad \ldots\ldots \quad (c_6).*$$

It may be of some use, in forming practical notions, to briefly advert to the geometrical signification of the set of formulæ just given. In the diagram, as before, the given series of quantities V_0, V_1, V_2, &c., are represented by the darker ordinates, the upper extremities of which are considered to be sufficient to determine the contour of the curve line P P_1 P_2 &c. If these extremities be also connected by straight lines, PP_1, P_1P_2, P_2P_3, &c., a polygonal figure will be formed. Suppose that $m=5$, $n=6$, or that four ordinates are



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* An expression equivalent to the first part of this last formula, for finding the area of a figure, was given by the late Thomas Weddle, in the *Cambridge and Dublin Mathematical Journal*, New Series, vol. ix., p. 79.

to be interpolated in each of six intervals, as shown in the diagram. Then, if the small subdivisions (Mm) of the line of abscissæ be taken as unity, we shall have $MM_1 = m$, and the total abscissa or base line $MN = mn$. Also, if we consider the polygonal figure bounded by the extreme ordinates and the base, we shall have

$$\text{Polygonal area} = m\{(V + V_1 \ldots . + V_n) - \tfrac{1}{2}(V_0 + V_n)\}.$$

Consequently, the corresponding average of the given quantities $V_0, V_1, \ldots . V_n$, estimated as before described, is

$$\frac{(V_0 + V_1 \ldots . + V_n) - \frac{1}{2}(V_0 + V_n)}{n} = \frac{\text{Polygonal area}}{\text{Base}}.$$

Again, if after the insertion of the interpolated ordinates the upper extremities of the entire set be consecutively joined, an interpolated polygonal figure will be formed, the area of which will be expressed by the sum of all the ordinates after deducting one-half of the extreme values, that is, by $\Sigma(V')_{0 \ldots n}$;

$$\therefore \frac{\Sigma(V')_{0 \ldots . n}}{mn} = \frac{\text{Interpolated polygonal area}}{\text{Base}}.$$

It is hence evident that, if the formula (C) be multiplied by the base $MN (= mn)$, the respective terms of the product will be thus geometrically represented:—

Interpolated polygon = Primitive polygon
− Curvilinear segments
+ Interpolated segments.

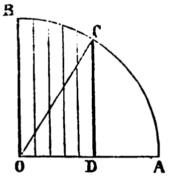
Also, if the particular formulæ (c_2), . . . (c_6), be severally multiplied by the base $MN (= mn)$, the geometrical signification of the terms will become

Interpolated polygon = Curvilinear area
+ Interpolated segments.

In these relations, it will not pass unnoticed that the several segments must be considered as positive or negative, according as the convexity of the curve may happen to be presented downwards or upwards.

It may be further observed, that the term which expresses the segments contains the reciprocal of m^2 as a factor; from which we draw the practical conclusion, that if a number of small segments of nearly equal length be cut off from a given curvilinear area, so as to reduce it to a polygon, the value of the sum of these segments will vary inversely as the square of their number.

By way of showing the degree of accuracy of the formulæ (c_5), (c_6), we shall employ them in finding the quadrature of the circle.



Let CD be an ordinate bisecting the radius OA of the quadrant AOB, and suppose four additional equidistant ordinates to be interposed between BO and CD. To obviate fractions, take OA = OB = 10; then OD = 5, and the six abscissæ, estimated from O, will be respectively 0, 1, 2, 3, 4, 5. The corresponding ordinates are, therefore, $V_0 = 10$, $V_1 = \sqrt{100-1} = 3\sqrt{11}$, $V_2 = \sqrt{100-4} = 4\sqrt{6}$, $V_3 = \sqrt{100-9} = \sqrt{91}$, $V_4 = \sqrt{100-16} = 2\sqrt{21}$, and $V_5 = \sqrt{100-25} = 5\sqrt{3}$. Hence, by the first part of the formula (c_5), the mean ordinate of the curvilinear area BD is found to be

$$(V) = \frac{5}{288}\left\{19(2+\sqrt{3}) + 15(3\sqrt{11}+2\sqrt{21}) + 10(4\sqrt{6}+\sqrt{91})\right\}.$$

Also the area BODC = OD × (V), and the area of the triangle COD = OD × $\frac{1}{2}$CD = OD × $\frac{1}{2}V_5$; ∴ the sector BOC = OD$\left\{(V) - \frac{1}{2}V_5\right\}$. Now, as BC is an arc of 30°, the area of the quadrant AOB is three times this sector, and if we divide by $r^2 = 100$, to reduce the scale of measurement to radius = unity, the result is

$$\begin{aligned}\frac{3}{20}\left\{(V) - \tfrac{1}{2}V_5\right\} &= \frac{38+5(2\sqrt{91}+8\sqrt{6}+6\sqrt{21}+9\sqrt{11}-25\sqrt{3})}{384} \\ &= \frac{38+5(\sqrt{364}+\sqrt{384}+\sqrt{756}+\sqrt{891}-\sqrt{1875})}{6.8.8} \\ &= \cdot 7853973,\end{aligned}$$

which is very easily found from a table of square roots. This computation differs from the true value (·7853982 &c.) by only an unit in the sixth place of decimals, and gives a nearer approximation to the circle than would be obtained by taking an inscribed polygon of 2,000 sides.

To determine the same area from seven ordinates, or six intervals, it will be requisite to interpose five ordinates between BO and CD; and fractions will be avoided by first taking OA = OB = 12. Thus, OD = 6, and the seven ordinates are respectively

$$\begin{array}{llllll} V_0 & = & \ldots\ldots\ldots & & = & 12\cdot0000000 \\ V_1 & = & \sqrt{144-1} & = \sqrt{143} & = & 11\cdot9582607 \\ V_2 & = & \sqrt{144-4} & = \sqrt{140} & = & 11\cdot8321596 \\ V_3 & = & \sqrt{144-9} & = \sqrt{135} & = & 11\cdot6189500 \\ V_4 & = & \sqrt{144-16} & = \sqrt{128} & = & 11\cdot3137085 \\ V_5 & = & \sqrt{144-25} & = \sqrt{119} & = & 10\cdot9087121 \\ V_6 & = & \sqrt{144-36} & = \sqrt{108} & = & 10\cdot3923048 \end{array}$$

$$\begin{array}{rcl} \therefore\ V_0+V_6 & = & 22\cdot3923048 \\ V_1+V_5 & = & 22\cdot8669728 \\ V_2+V_4 & = & 23\cdot1458681 \end{array}$$

and by substitution in the first part of the formula (c_6) the mean ordinate is

$$(\mathrm{V})=11{\cdot}4793368.$$

Also, as before, the sector BOC=OD$\{(\mathrm{V})-\frac{1}{2}\mathrm{V}_6\}$, which, multiplied by 3 and divided by $r^2=144$, the area of a quadrant, to radius=unity, comes out ·7853981, which is only an unit in error in the seventh place of decimals, and no part of the calculation has been taken beyond that place.*

Having completed the theory of summation, as originally proposed, and endeavoured to elucidate descriptively and practically the principles upon which it rests, we shall now take up the subject in a different manner, and show how the general expression (5), p. 307, may be transformed into a new formula, which, after some slight modification, is applicable, in many cases, to a yet more simple and easy calculation of areas or volumes, and is also most especially adapted to expedite by a brief and elementary process the computation of the values of annuities that involve the contingencies of any number of lives.

Restoring the values of the coefficients in terms of m, the equation (5) gives

$$\begin{aligned}\frac{\Sigma(\mathrm{V}')_{0\ldots.n}}{m}=\Sigma(\mathrm{V})&+\frac{m^2-1}{12m^2}\Big\{(a_0)-(a_n)\Big\}\\ &-\frac{(m^2-1)(11m^2+1)}{720m^4}\Big\{(c_0)-(c_n)\Big\}\\ &+\frac{(m^2-1)(191m^4+23m^2+2}{60480m^6}\Big\{(e_0)-(e_n)\Big\}\\ &\qquad\&c.\end{aligned}$$

$$=\Sigma(\mathrm{V})$$

$$+\left\{\frac{m^2-1}{12m^2}(a_0)-\frac{11m^4-10m^2-1}{720m^4}(c_0)+\frac{191m^6-168m^4-21m^2-2}{60480m^6}(e_0)-\&c.\right\}$$

$$-\left\{\frac{m^2-1}{12m^2}(a_n)-\frac{11m^4-10m^2-1}{720m^4}(c_n)+\frac{191m^6-168m^4-21m^2-2}{60480m^6}(e_n)-\&c.\right\}.$$

Now, if we suppose a value V_x to be taken near to V_0, and divide the increment $\mathrm{V}_x-\mathrm{V}_0$ by x, the quotient, after x is made

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[*] For the ratio of the circumference of a circle to its diameter I have, by means of an empirical process, constructed the concise numerical expression

$$3{\cdot}1+\frac{1}{4}\left\{\frac{1}{6}-\frac{{\cdot}09}{304}\right\},$$

which is extremely simple considering the degree of accuracy attained. It is correct to no less than ten decimal places of figures.

$=0$, will express the value of the differential ratio $\frac{dV}{dx}$ at that point. Hence, from the expression (a) in terms of the central differences (article on "Interpolation," p. 68) we thus find, at the initial point,

$$\left(\frac{dV}{dx}\right)_0=\left(\frac{V_x-V_0}{x}\right)_0=(a_0)-\frac{1}{2.3}(c_0)+\frac{1.4}{2.3.4.5}(e_0)-\&c. \quad . \; . \; . \quad (8)$$

Similarly at the terminal point, where $x=n$, we shall have

$$\left(\frac{dV}{dx}\right)_n=(a_n)-\frac{1}{2.3}(c_n)+\frac{1.4}{2.3.4.5}(e_n)-\&c.$$

Therefore, by substitution, we get

$$*\,\frac{\Sigma(V')_{0\ldots\ldots n}}{m}=\Sigma(V)+\frac{m^2-1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_0-\left(\frac{dV}{dx}\right)_n\right\}$$
$$-\frac{m^4-1}{720m^4}\left\{(c_0)-(c_n)\right\}+\frac{23m^6-21m^2-2}{60480m^6}\left\{(e_0)-(e_n)\right\}-\&c.,$$

where the latter terms, being respectively less than $\frac{1}{720}\left\{(c_0)-(c_n)\right\}$, $\frac{1}{2630}\left\{(e_0)-(e_n)\right\}$, &c., may be rejected as insignificant.

The expression thus reduces down to the brief and neat form

$$\left.\begin{aligned}\frac{\Sigma(V')_{0\ldots\ldots n}}{m}&=\Sigma(V)+\frac{m^2-1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_0-\left(\frac{dV}{dx}\right)_n\right\}\\&=\Sigma(V)+\frac{m^2-1}{12m^2}(\tan\theta_0-\tan\theta_n)\end{aligned}\right\} \quad . \; . \quad (d),$$

where θ denotes the angle which a tangent to the curve makes with the line of abscissæ. As the interval between the interpolated ordinates is $\frac{1}{m}$, this expresses the area of the interpolated polygon, and the first term $\Sigma(V)$ expresses that of the primitive polygon.

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* By successively differentiating the expression (a) referred to, and employing the initial values of the odd differential coefficients, that is, the values when $x=0$, the formula may be completely resolved into the following remarkable development:—

$$\begin{aligned}\frac{\Sigma(V')_{0\ldots\ldots n}}{m}=\Sigma(V)&+\frac{m^2-1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_0-\left(\frac{dV}{dx}\right)_n\right\}\\&-\frac{m^4-1}{720m^4}\left\{\left(\frac{d^3V}{dx^3}\right)_0-\left(\frac{d^3V}{dx^3}\right)_n\right\}\\&+\frac{m^6-1}{30240m^6}\left\{\left(\frac{d^5V}{dx^5}\right)_0-\left(\frac{d^5V}{dx^5}\right)_n\right\}\\&-\quad\&c. \qquad\qquad \&c.\end{aligned}$$

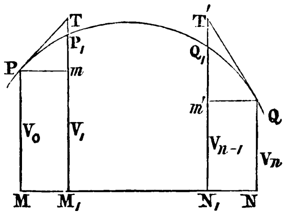
Also, by making the number m indefinitely great, the curvilinear area is

$$\left.\begin{aligned} &=\Sigma(\mathrm{V})+\frac{1}{12}\left\{\left(\frac{d\mathrm{V}}{dx}\right)_0-\left(\frac{d\mathrm{V}}{dx}\right)_n\right\} \\ &=\Sigma(\mathrm{V})+\frac{1}{12}(\tan\theta_0-\tan\theta_n) \end{aligned}\right\} \quad . \quad (d').^*$$

And from these expressions we further conclude, that the sum of the curvilinear segments $=\frac{1}{12}(\tan\theta_0-\tan\theta_n)$, and the sum of the interpolated segments $=\frac{1}{12m^2}(\tan\theta_0-\tan\theta_n)$, it being remembered that $\tan\theta$ is positive when the curve ascends, and negative when it descends in a forward direction.

The formulæ (d) and (d') are remarkably simple, and suggest the following geometrical statement:—

In the diagram P, Q, are the initial and terminal points of the curve; V_0, V_1, are the first two, and V_{n-1}, V_n, the last two of the given ordinates. Let tangents PT, QT′, be drawn touching the curve at the two extreme points, P, Q, meeting the adjacent ordinates, V_1, V_{n-1}, produced if necessary, in T, T′; and let Pm, Qm', be drawn parallel to the horizontal line of abscissæ, meeting the same ordinates in m, m'. Then (P) denoting the polygonal area $\mathrm{MPP}_1 \ldots\ldots \mathrm{Q}_1\mathrm{QN}$, and the intervening distance MM_1 being not necessarily considered as the linear unit, we shall have



$$\text{Polygonal area (P)}=\Sigma(\mathrm{V}).\mathrm{MM}_1$$
$$=\{(\mathrm{V}_0+\mathrm{V}_1\ldots\ldots+\mathrm{V}_n)-\tfrac{1}{2}(\mathrm{V}_0+\mathrm{V}_n)\}.\mathrm{MM}_1.$$

After $m-1$ ordinates are interposed between each of the n intervals,

$$\text{Interpolated polygon }(\mathrm{P}')=\Sigma(\mathrm{V}').\frac{\mathrm{MM}_1}{m}$$
$$=(\mathrm{P})+\frac{m^2-1}{12m^2}(\mathrm{T}m+\mathrm{T}'m').\mathrm{MM}_1$$
$$=(\mathrm{P})+\frac{m^2-1}{6m^2}(\mathrm{T}+\mathrm{T}').$$

---

* Here the interval or distance between the given ordinates is supposed to be unity. If the distance be a the curvilinear area will be

$$=a.\Sigma(\mathrm{V})+\frac{a^2}{12}(\tan\theta_0-\tan\theta_n).$$

And, after the interpolation of an indefinitely great number of ordinates,

$$\text{Curvilinear area} = (\mathrm{P}) + \frac{1}{12}(\mathrm{T}m + \mathrm{T}'m').\mathrm{MM}_1$$

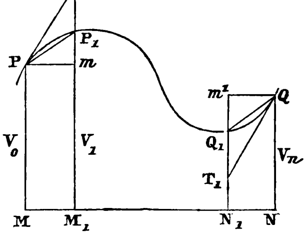
$$= (\mathrm{P}) + \frac{1}{6}(\mathrm{T}+\mathrm{T}'),$$

where T, T′, denote the respective areas of the tangential triangles, T$\mathrm{P}m$, T′$\mathrm{Q}m'$.

Hence, the sum of the areas of the curvilinear segments cut off by the sides of the primitive polygon $= \frac{\mathrm{T}+\mathrm{T}'}{6}$, and the sum of those cut off by the sides of the interpolated polygon, $= \frac{\mathrm{T}+\mathrm{T}'}{6m^2}$. The lines T$m$ T′m', and the tangential triangles T, T′, must, of course, be accounted as negative in value whenever the points T, T′, may happen to fall below instead of above the horizontal lines which pass through the extreme points P, Q. They are, in fact, positive or negative, according as the tangents ascend or descend from the point of contact.

From the generality of the investigation by which these results have been obtained, they are evidently independent of the existence of any law of sinuosity or flexure. The only essential requisitions are, continuity of outline, a nonvertical position at the extreme points, and a sufficiently narrow interval between the ordinates to include peculiarities. That is, with the knowledge of the initial and terminal directions, PT, QT′, the intermediate points, P_1, P_2, &c., should be sufficiently numerous to indicate the track of the curve.

A somewhat curious conclusion may be drawn, if we conceive the case of a curve having contrary flexures, which bring its course back again to its primitive direction, as, for example, in the annexed diagram. The tangents, PT, QT_1, are here parallel, and the tangential triangles are equal in magnitude but contrary in algebraic sign, so that their sum is zero. Hence, in this particular case, the polygonal and curvilinear areas are equal. This result is, in some measure, explained by the apparent fact that, according as the convexity of



the curve is presented upwards or downwards, the curvilinear segments are severally added to or cut off from the polygonal area; and it is remarkable that these alternations, when taken in combination, should become practically balanced.

Independently of the mathematical investigation, a little consideration will elicit a popular explanation of the philosophical nature of these geometrical deductions. Thus, when the curve has but little curvature, and alters its course gradually, the segments cut off by the chords are severally smaller in magnitude, but as the curve is longer in effecting a given change in its direction, a greater number of segments will necessarily intervene. On the other hand, when the curve has more curvature, and turns more suddenly, it will be perceived that the segments are comparatively greater in magnitude, but, as the curve sooner acquires its new direction, they are fewer in number. In the two cases, therefore, when the values are estimated between given tangential directions, the sum of the segments may be regarded as having practically the same value.

In order to adapt the formula to the computation of life annuities, let the unit of interval be taken between the interpolated instead of the primitive values, which will now be, V_0, V_m, V_{2m}, ... V_n. Also as an interval of time (t) expressed in years or parts of a year, is measured accordingly, the symbol x, denoting the abscissa, should obviously be replaced by $\frac{t}{m}$. Thus, the formula (d) becomes

$$\Sigma(V')_{0\dots n}=m\Sigma(V)+\frac{m^2-1}{12}\left\{\left(\frac{dV}{dt}\right)_0-\left(\frac{dV}{dt}\right)_n\right\}.$$

But when the annuity is completed, and the series representing it is entirely run out, the final number will extend beyond the limiting or highest age of the table of mortality, and we shall then evidently have $V_n=0$, $\left(\frac{dV}{dt}\right)_n=0$;

$$\therefore\ \Sigma(V')=m\Sigma(V)+\frac{m^2-1}{12}\left(\frac{dV}{dt}\right)_0.$$

Let now ρ denote the yearly interest on £1;

v the present value of £1 certain receivable a year hence;

l_0 the number living at the given age according to the table of mortality;

l or l_t the number living after t years have elapsed;

V or V_t the present value of the tth annual payment of £1 receivable by the annuitant on the expiration of t years;

then $$V=\frac{l}{l_0}v^t$$

$$^*\log V=\log l+t\log v-\log l_0;$$

and, differentiating with respect to t,

$$\frac{1}{V}\cdot\frac{dV}{dt}=\frac{1}{l}\cdot\frac{dl}{dt}+\log v$$
$$=-\mu-\log(1+\rho)$$
$$=-(\mu+\delta),$$

where μ denotes the *force of mortality* at the given age, and δ the rate of interest when it is assumed to be payable momently. The latter quantity (δ), as it plays an analagous and equally important part in the effective diminution of the value of V, we shall, with like propriety, designate the *force of discount.*

Now, if A denote the required value of the annuity, and we calculate

$$S=V_m+V_{2m}+V_{3m}+\&c.,$$

we shall evidently have

$$\Sigma(V')=\tfrac{1}{2}V_0+(V_1+V_2+V_3+\&c.)$$
$$=\tfrac{1}{2}+A$$
$$(V)=\tfrac{1}{2}V_0+(V_m+V_{2m}+V_{3m}+\&c.)$$
$$=\tfrac{1}{2}+S.$$

By substitution, the formula hence becomes

$$\tfrac{1}{2}+A=m(\tfrac{1}{2}+S)-\frac{m^2-1}{12}(\mu+\delta)$$

$$\therefore\ A=mS+\frac{m-1}{2}-\frac{m^2-1}{12}(\mu+\delta)\quad .\ .\ .\quad (D),$$

and it will be perceived that S is the present value of an annuity payable only after intervals of m years, which, when m is taken at a wide interval, will consist of very few terms, and involve comparatively but little calculation.

It may be observed as the peculiar and valuable feature of the formula (D), that, irrespective of the number m, the data

$$V_0=1,\ \left(\frac{dV}{dt}\right)_0=-(\mu+\delta)$$

correctly fix the initial position and direction of the annuity curve. Thus, when we have to consider the assumption of the number m, in connexion with the important adjustment alluded to, it will appear that the subsistence of very few of the distributed ordinates,

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* In mathematical analysis hyp. log. is always understood, the common logarithm being exclusively employed as a subsidiary in numerical calculations.

V_m, V_{2m}, V_{3m}, &c., will be sufficient to impart to the curve its specific form and character, and thereby ensure a requisite degree of accuracy in the result of the computation.

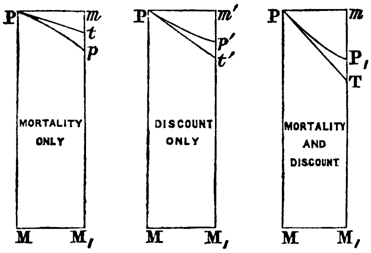
The formula will apply equally to a joint annuity on any number of lives, if only $\mu+\mu'+\&c.$, be substituted in place of μ, according to the following theorem:—

THEOREM.—*The force of mortality of a joint existence is equal to the sum of the forces of mortality of the constituent lives.*

The demonstration may be readily inferred from the differential relation

$$\frac{d(ll'\ \&c.)}{ll'\ \&c.}=\frac{dl}{l}+\frac{dl'}{l'}+\&c.$$

The precise adaptation of this theorem to the method before us leaves nothing to be desired, as it at once brings the calculation of a joint annuity on any number of lives under the same elementary process as that for a single life. We may here briefly state the corresponding geometrical relations.



If, in the first diagram, the ordinates PM, pM$_1$, &c., be supposed to represent the proportion of survivors after 0, 1, &c., years, the curve Pp, &c., will be the curve of mortality; and if Pm be drawn parallel to MM$_1$ and Pt a tangent to the curve at P, meeting the second ordinate pM$_1$ produced in m and t, then will mt represent μ the force of mortality at P; and if the curve of mortality be that of several lives jointly, it will, by the theorem, be the sum of the individual values of μ.

Again, in the second diagram, if the ordinates, PM, p'M$_1$, &c., be supposed to represent 1, v, v^2, &c., the present values of the payments of an annuity certain, then the logarithmic curve Pp', &c., will be that of discount, and it will differ from the former by its convexity being uniformly presented downwards. Drawing the lines Pm', Pt', as before, $m't'$ will represent the value of δ the force of discount, which is invariable and independent of age.

Lastly, in the third diagram, if the ordinates represent the values of the proportion of the survivors discounted for present values, the curve PP$_1$, &c., will be that of the annuity payments at the proposed rate of interest, and mT the total force of decrement will, according to the theorem, be the sum of those of mortality and discount, shown in the two former diagrams.

In the several diagrams, only one yearly interval is exhibited, the others not being required for our present object. It will be observed, that in each case the force of mortality, discount, or of total decrement is the dip or vertical descent of the initial tangent when it has reached the next yearly ordinate.

Resuming now the general case of a joint annuity, if p_t, p'_t, &c., be the probabilities of the respective lives surviving t years, $p_t p'_t$ &c., or $(pp' \text{ \&c.})_t$ will express the probability of their joint existence after t years. Thus we shall have

$$\begin{aligned} &\mathrm{V}_m = (pp' \text{ \&c.})_m . v^m \\ &\mathrm{V}_{2m} = (pp' \text{ \&c.})_{2m} . v^{2m} \\ &\text{\&c.} \qquad\qquad \text{\&c.} \end{aligned}$$

$$\mathrm{A} = m(\mathrm{V}_m + \mathrm{V}_{2m} + \text{\&c.}) + \frac{m-1}{2} - \frac{m^2-1}{12}(\mu + \mu' + \text{\&c.} + \delta) \quad . \; . \quad (\mathrm{D}').$$

It is another material advantage in the present method that the formula may be employed with equal facility and without any additional labour, when the joint annuity is held on a number of lives which, taken separately, are classed under different tables of mortality. We have merely to take the probabilities of survivorship (p, p', &c.) and the forces of mortality (μ, μ', &c.) from the respective tables, and the form and extent of the calculation is precisely the same.

The interval m may, in most cases, be extended up to about 20 years. In general, however, the best intervals to adopt will be 5, 7, or 11 years. For these intervals, the coefficients, &c., are convenient integers, and the special formulæ are

$$\begin{aligned} m &= 5 \text{ years}, & \mathrm{A} &= 5\mathrm{S} + 2 - 2(\mu + \delta) \\ m &= 7 \quad ,, & \mathrm{A} &= 7\mathrm{S} + 3 - 4(\mu + \delta) \\ m &= 11 \quad ,, & \mathrm{A} &= 11\mathrm{S} + 5 - 10(\mu + \delta). \end{aligned}$$

To practically elucidate the method, tables are here given showing the values of the force of mortality (μ) at each age, according to the Experience, Davies' Equitable, and Milne's Carlisle tables, and the force of discount (δ) at several rates of interest.

The values of the force of discount have been calculated from the formula $\delta = \text{hyp. log}\,(1+\rho)$. It is nearly equal to $\frac{1}{2}(1 - v + \rho)$, that is, the average between the discount and interest of £1 for one year, and it is in reality that rate of interest which, if accumulated momently, would be equivalent to that which actually exists.

The force of mortality has been found by dividing half the sum of the decrements immediately preceding and following by the number living at the given age. The formula (8) might have been made use of as the correct numerator, but with small numbers no increase of accuracy would be gained. Any irregularities of progression must be attributed to the imperfect adjustment of the original tables.

Force of Mortality (μ).

Age.	Experience.	Milne's Carlisle.	Davies' Equitable.	Age.	Experience.	Milne's Carlisle.	Davies' Equitable.
10	·00678	·0048	·0039	51	·01655	·0139	·0189
11	·00680	·0047	·0039	52	·01757	·0149	·0204
12	·00682	·0049	·0041	53	·01868	·0158	·0219
13	·00685	·0051	·0043	54	·01989	·0167	·0230
14	·00690	·0054	·0045	55	·02120	·0176	·0238
15	·00695	·0059	·0048	56	·02263	·0186	·0250
16	·00700	·0065	·0052	57	·02418	·0201	·0262
17	·00706	·0068	·0056	58	·02585	·0228	·0276
18	·00712	·0070	·0060	59	·02767	·0265	·0290
19	·00719	·0070	·0064	60	·02970	·0313	·0302
20	·00727	·0071	·0066	61	·03195	·0352	·0311
21	·00736	·0070	·0067	62	·03442	·0373	·0325
22	·00745	·0070	·0069	63	·03712	·0386	·0343
23	·00754	·0070	·0072	64	·04008	·0398	·0363
24	·00764	·0071	·0074	65	·04332	·0413	·0384
25	·00779	·0072	·0077	66	·04686	·0427	·0408
26	·00786	·0074	·0079	67	·05073	·0444	·0434
27	·00798	·0076	·0084	68	·05495	·0464	·0463
28	·00810	·0083	·0088	69	·05950	·0489	·0494
29	·00824	·0093	·0093	70	·06443	·0516	·0525
30	·00838	·0100	·0096	71	·06980	·0567	·0560
31	·00854	·0102	·0099	72	·07563	·0653	·0598
32	·00870	·0102	·0104	73	·08195	·0756	·0642
33	·00887	·0101	·0107	74	·08882	·0874	·0693
34	·00905	·0102	·0108	75	·09631	·0973	·0738
35	·00923	·0103	·0112	76	·10442	·1043	·0789
36	·00943	·0105	·0115	77	·11326	·1111	·0849
37	·00963	·0108	·0118	78	·12295	·1146	·0918
38	·00984	·0111	·0122	79	·13350	·1203	·1011
39	·01007	·0116	·0124	80	·14496	·1280	·1125
40	·01030	·0125	·0125	81	·15739	·1362	·1256
41	·01054	·0135	·0127	82	·17083	·1476	·1408
42	·01081	·0142	·0128	83	·18546	·1573	·1589
43	·01113	·0146	·0132	84	·20157	·1682	·1790
44	·01154	·0148	·0139	85	·21958	·1820	·2009
45	·01202	·0149	·0143	86	·24025	·2030	·2238
46	·01260	·0149	·0145	87	·26419	·2280	·2448
47	·01326	·0148	·0150	88	·29247	·2478	·2613
48	·01398	·0144	·0157	89	·32672	·2486	·2706
49	·01476	·0139	·0165	90	·36847	·2676	·2769
50	·01562	·0136	·0176				

Force of Discount (δ).

Interest per Cent.	1	·00995	Interest per Cent.	3	·02956	Interest per Cent.	5	·04879
	$1\frac{1}{2}$	·01489		$3\frac{1}{2}$	·03440		6	·05827
	2	·01980		4	·03922		7	·06766
	$2\frac{1}{2}$	·02469		$4\frac{1}{2}$	·04402		8	·07696

Example 1.—Required the present value of an annuity payable during the existence of a life aged 40, according to the experience table of mortality and interest at the rate of $3\frac{1}{2}$ per cent.

Take $m=7$ years, and from $V_t=p_t.v^t$ we find

$$\begin{array}{lr}
 & V_7 \;=\; \cdot7253 \\
 & V_{14} = \cdot5089 \\
 & V_{21} = \cdot3351 \\
 & V_{28} = \cdot1959 \\
 & V_{35} = 919 \\
 & V_{42} = 291 \\
 & V_{49} = 44 \\
 & V_{56} = 1 \\ \cline{2-2}
 & S\ldots.1\cdot8907 \\
 & 7 \\ \cline{2-2}
 & 13\cdot2349 \\
\mu\ \cdot01030 & +3 \\ \cline{2-2}
\delta\ 3440 & 16\cdot2349 \\ \cline{1-1}
\cdot04470 \times -4= & -\cdot1788 \\ \cline{2-2}
 & A\ldots.16\cdot0561
\end{array}$$

Otherwise, take $m=11$ years, then

$$\begin{array}{lr}
 & V_{11} = \cdot5957 \\
 & V_{22} = \cdot3132 \\
 & V_{33} = \cdot1177 \\
 & V_{44} = 187 \\
 & V_{55} = 2 \\ \cline{2-2}
 & S\ldots.1\cdot0455 \\
 & 11 \\ \cline{2-2}
 & 11\cdot5005 \\
\mu\ \cdot01030 & +5 \\ \cline{2-2}
\delta\ 3440 & 16\cdot5005 \\ \cline{1-1}
\cdot04470 \times -10= & -\cdot4470 \\ \cline{2-2}
 & A\ldots.16\cdot0535
\end{array}$$

Jenkin Jones, in his Experience Tables, entitled *New Rate of Mortality*, gives the value 16·055.

Example 2.—Required the value of a joint annuity on three lives, aged 35, 60, 65, according to the Carlisle table, at 5 per cent.

Take $m=5$, then from $V_t=(pp'p'')_t.v^t$ we find*

$$\begin{array}{lcr}
V_5 & = & \cdot4888 \\
V_{10} & = & \cdot1980 \\
V_{15} & = & 573 \\
V_{20} & = & 110 \\
V_{25} & = & 12 \\
V_{30} & = & 1 \\
V_{35} & = & 0 \\
\hline
\end{array}$$

$$\begin{array}{lr@{\qquad}r}
\mu & \cdot0103 & \text{S}\ldots\cdot7564 \\
\mu' & 313 & 5 \\
\mu'' & 413 & \overline{} \\
 & \overline{} & 3\cdot7820 \\
 & \cdot0829 & +2 \\
\delta\ldots & 488 & \overline{} \\
 & \overline{} & 5\cdot7820 \\
 & \cdot1317\times-2= & -\cdot2634 \\
 & & \overline{} \\
 & & \text{A}\ldots 5\cdot5186
\end{array}$$

Milne, in his *Treatise on Annuities and Assurances*, Table L., p. 676, gives the value 5·519.

By means of the formula (d), we can deduce the correction that should be made in the value of a life annuity when each yearly payment is made in m instalments. Let V_1, V_2, V_3, &c., be the present values of £1 receivable by the life, respectively after 1, 2, 3, &c., years, so that

$$A=V_1+V_2+V_3+\&c.,$$

and the value of the annuity, when each £1 is payable in m instalments of $\frac{1}{m}$ each, will be

$$A'=\frac{1}{m}(V'_1+V'_2+V'_3+\&c.)$$

Also, since $V_0=1$, we shall evidently have $\Sigma(V)=\frac{1}{2}+A$ and $\Sigma(V')=\frac{1}{2}+mA'$; and, beyond the limiting age of the table, $\frac{dV}{dt}=0$. Therefore (d) gives

$$\frac{\frac{1}{2}+mA'}{m}=\frac{1}{2}+A+\frac{m^2-1}{12m^2}\left(\frac{dV}{dt}\right)_0$$

$$=\frac{1}{2}+A-\frac{m^2-1}{12m^2}(\mu+\delta)$$

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* The values of $\log p_t$ are easily got out from a table of $\log l$, by placing a card over $\log l_0$ and another under $\log l_t$, so as to obtain their difference without transcribing them; and if for any one of the lives a table of $\log D$ is available in place of $\log l$, the factor v^t will be thereby included, which will so far abbreviate the calculation. The logarithms taken to five places will be quite sufficient.

$$\therefore A' = A + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu+\delta) \quad . \quad . \quad . \ (9)$$

where the positive term is the polygonal correction, and the negative term the correction for curvature, the latter of which has generally been disregarded.

Hence, when a life annuity is payable

$$\left.\begin{array}{ll} \text{Half-yearly,} & A' = A + \frac{1}{4} - \frac{1}{16}(\mu+\delta) \\ \text{Quarterly,} & A' = A + \frac{3}{8} - \frac{5}{64}(\mu+\delta) \\ \text{Momently,} & A' = A + \frac{1}{2} - \frac{1}{12}(\mu+\delta) \end{array}\right\} \quad . \quad . \quad . \quad . \ (10).$$

The present value of a life premium of £1 payable yearly, being that of an annuity payable in advance, $= 1 + A$. Add an advance payment $= \frac{1}{m}$ to (9) and the present value of a life premium of £1 per annum, payable in m instalments, is

$$A + \frac{m+1}{2m} - \frac{m^2-1}{12m^2}(\mu+\delta)$$

$$= (1+A) - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu+\delta).$$

Therefore if π denote the ordinary annual premium for a life assurance, the same when payable in m instalments will be

$$\pi' = \frac{\pi(1+A)}{(1+A) - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu+\delta)}$$

$$= \frac{\pi}{1 - (1 - v + \pi)\left(\frac{m-1}{2m} + \frac{m^2-1}{12m^2}(\mu+\delta)\right)} \quad . \quad . \ (11).$$

According to the formulæ hitherto given, the value is somewhat underrated, more especially at the higher ages.*

The formula (d') will also serve to determine, from a given table of mortality, the mean duration of life after having attained a given age. This is more commonly called the "expectation of life," and, in the tables usually given in works on life contingencies, it is calculated on the supposition that the annual decrements are equally distributed throughout each year. As this assumption is, in fact, equivalent to the substitution of a polygon in place of the true curve of mortality, it must evidently involve some error, and in summing the terms this must accumulate from year to year. If, in the mortality table, l_0, l_1, l_2, &c., denote the successive numbers living, commencing with the given age, and these be taken as ordinates and the number of years elapsed as the corresponding abscissæ of the curve, then the formula (d') will express ($\int l dt$) the total number of years of life eventually passed through by the persons originally represented by the number l_0. Hence, observing that at the end of the table $\frac{dl}{dt}=0$, and dividing by l_0, we find the mean duration of life to be

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* In official practice it is a safe and convenient rule to take the yearly premium a year higher when it is to be paid quarterly, and half a year higher when it is to be paid half-yearly. This will rather more than cover the true calculation, as it should do to provide for the cost of additional receipts, &c.

$$=\frac{\Sigma(l)}{l_0}+\frac{1}{12}\cdot\frac{1}{l_0}\left(\frac{dl}{dt}\right)_0$$

$$=\tfrac{1}{2}+\frac{l_1+l_2+l_3+\&c.}{l_0}-\frac{\mu}{12}\quad .\ .\ .\ .\quad (12).$$

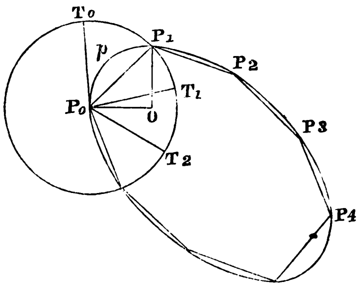
It is, in fact, the same as the value of an annuity payable momently, and supposing money to have no interest. Thus it appears that the tabular expectations of life, usually given, are in excess of the truth by one-twelfth of the force of mortality at the respective ages. The values of this force, of which a table has been given, will show that the requisite correction is decidedly appreciable at the advanced ages, but the error is not of much consequence, as the mean duration of life is an element that is only made use of in cursory comparisons, and is not required in professional calculations.

Reverting, for an instant, to the application of the general subject of investigation, we have found it to resolve itself into the determination of a convenient practical relation expressing with sufficient accuracy the value of the segments which make up the difference between a curvilinear area and that of an inscribed polygonal figure, the points of which are marked out by a given series of equidistant ordinates. Before closing this article, it may not be uninteresting to subjoin a rather neat geometrical representation of the corresponding relation when the several sides of the inscribed polygon are equal in length.

Let $P_0P_1P_2$, &c., be a polygon of equal sides inscribed in a curve, and, taking P_0 as origin, let the length of the arc of the curve, to any point p, $=s$; the angle which the tangent to the curve makes with the initial tangent $=\theta$; the radius of curvature, in the vicinity of the point p, $=\mathrm{R}$; and the chord $\mathrm{P}_0\mathrm{P}_1$, or side of the polygon, $=c$. Then by disregarding, as inappreciable, certain small terms of the fifth and higher orders, it may be shown, by the differential calculus, that the curvilinear segment between P_0 and P_1 may be estimated as equivalent to that of a circle having the average curvature between those points. Now, if ds, $d\theta$, denote corresponding small increments at the point p, we shall evidently have $\mathrm{R}d\theta=ds$, or $\mathrm{R}=\frac{ds}{d\theta}$; and therefore the average value of R, for all points of the curve lying between P_0 and P_1, is

$$(\mathrm{R})=\frac{\delta s}{\delta\theta},$$

where δs, $\delta\theta$, are the values estimated between the extreme points of the segment.



Hence, treating the segment of the curve from P_0 to P_1 as equivalent to that of a circle whose centre is 0 and radius $=(\mathrm{R})$, and observing that the angle $\mathrm{P}_0 0 \mathrm{P}_1=\delta\theta$, we shall have

$$\begin{aligned}
c^2=(\mathrm{R})^2\mathrm{ch} \quad &=2(\mathrm{R})^2(1-\cos\delta\theta)\\
&=(\mathrm{R})^2\left(\delta\theta^2-\frac{\delta\theta^4}{3.4}+\frac{\delta\theta^6}{3.4.5.6}-\&\text{c.}\right)\\
&=\delta s^2\left(1-\frac{\delta\theta^2}{3.4}+\frac{\delta\theta^4}{3.4.5.6}-\&\text{c.}\right)
\end{aligned}$$

$$\begin{aligned}
\text{Also, sector }\ \mathrm{P}_0 0\mathrm{P}_1&=\tfrac{1}{2}(\mathrm{R})\delta s=\tfrac{1}{2}(\mathrm{R})^2\delta\theta\\
\text{triangle }\mathrm{P}_0 0\mathrm{P}_1&=\tfrac{1}{2}(\mathrm{R})^2\sin\delta\theta\\
\therefore\ \text{segment }\ \mathrm{P}_0\mathrm{P}_1&=\tfrac{1}{2}(\mathrm{R})^2(\delta\theta-\sin\delta\theta)\\
&=\tfrac{1}{2}(\mathrm{R})^2\left(\frac{\delta\theta^3}{2.3}-\frac{\delta\theta^5}{2..5}+\frac{\delta\theta^7}{2....7}-\&\text{c.}\right)\\
&=\frac{\delta s^2}{12}\delta\theta\left(1-\frac{\delta\theta^2}{4.5}+\frac{\delta\theta^4}{4.5.6.7}-\&\text{c.}\right)\ .\ .\ .\ (13)\\
&=\frac{c^2}{12}\delta\theta\left(1+\frac{\delta\theta^2}{5.6}+\frac{\delta\theta^4}{4.5.6.7}+\&\text{c.}\right)^*\ .\ .\ (14)\\
&=\frac{c^2}{12}\delta\theta\left(1+\frac{1}{\frac{30}{\delta\theta^2}-1}\right)\ .\ .\ .\ .\ .\ .\ .\ (15).
\end{aligned}$$

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* A further extension of this formula will show a remarkable approximation of the terms to a geometrical series.

In this last expression the reciprocal fraction within the parenthesis is so small that in the process of summation it may be considered as a constant depending on an average value of $\delta\theta$, which again depends on the total number of sides of the polygon. Thus, if n denote the number of sides, the average value of $\delta\theta=\frac{2\pi}{n}$ (where $\pi=3{\cdot}141592$, &c.); and $\therefore \frac{30}{\delta\theta^2}=\frac{30}{4\pi^2}.n^2=\cdot 76n^2$; and (3) hence becomes

$$\text{Segment}=\frac{c^2}{12}\delta\theta\left(1+\frac{1}{\cdot 76n^2-1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Therefore, summing for any number of consecutive segments, we get

$$(\Sigma\text{ segments})_{0\ldots\ldots m}=\frac{c^2}{12}\theta_m\left(1+\frac{1}{\cdot 76n^2-1}\right) \quad . \quad . \quad . \quad . \quad (17),$$

which is easy of calculation, and suggests the following compendious geometrical relations, which will hold generally, whatever changes may occur in the flexure of the curve.

With centre P_0 and radius equal to the side of the polygon describe a circle, and draw the tangential radius P_0T_0, which will indicate the direct course of the curve at the point P_0. Similarly, if we conceive tangents to be drawn to the curve at the remaining points, P_1, P_2, &c., and radii P_0T_1, P_0T_2, &c., to be drawn parallel to these tangents, the complete set of radii thus determined will respectively indicate the direct course pursued by the curve at the corresponding angles of the polygon.

Then, the respective areas of the curvilinear segments, P_0P_1, P_1P_2, &c., will be practically represented by one-sixth of the circular sectors contained between the successive radii P_0T_0, P_0T_1, P_0T_2, &c. Also, it follows that the sum of any number of consecutive curvilinear segments will be represented by one-sixth of the circular sector comprised between those radii which appertain to the extreme points of the series. And when the circuit of the curve is wholly included in the summation, the radius PT having then passed round the circumference of the circle and again reached its original position, the total sum of all the segments, cut off by the sides of the polygon, will be practically represented by one-sixth of the entire circle so described.

A higher degree of accuracy will obtain if the values thus exhibited be each of them increased by a small correction found by dividing it by the number $\cdot 76n^2-1$, or $\frac{3}{4}n^2-1$ nearly.

From the formula (13) and (14) it may be inferred that if the total length of the curve be divided into n equal portions, and a circle be described having the length of one of these portions as radius, then the total sum of the segments will be absolutely less in value than one-sixth of this circle, and greater than one-sixth of the circle whose radius is equal to the side of the polygon.

NOTE.—The general development given in the foot note of page 317 may be obtained by the process there indicated. The following is an independent investigation of it by means of the differential calculus.

To determine the value of the curvilinear segment which connects two consecutive ordinates V_0 and V_1, it will be convenient in the first instance to take the origin midway between them; then, by Maclaurin's theorem,

$$V = V_{\frac{1}{2}} + \left(\frac{dV}{dx}\right)x + \left(\frac{d^2V}{dx^2}\right)\frac{x^2}{2} + \left(\frac{d^3V}{dx^3}\right)\frac{x^3}{2\,.\,3} + \left(\frac{d^4V}{dx^4}\right)\frac{x^4}{2.3.4}\,\&c. \;\ldots\; (18)$$

in which the differential coefficients also apppertain to the middle point. If δx denote the intervening distance between consecutive ordinates, the values of V_0 and V_1 will be obtained by respectively putting $-\frac{1}{2}\delta x$ and $+\frac{1}{2}\delta x$ for x. Hence, taking the difference and half sum of the values thus found, we get

$$\delta V = \left(\frac{dV}{dx}\right)\delta x + \left(\frac{d^3V}{dx^3}\right)\frac{\delta x^3}{2.3.4} + \left(\frac{d^5V}{dx^5}\right)\frac{\delta x^5}{2\,.\,.\,5.4^2} + \left(\frac{d^7V}{dx^7}\right)\frac{\delta x^7}{2\,.\,.\,.\,.\,7.4^3}\,\&c.\,.\; (19)$$

$$(V) = V_{\frac{1}{2}} + \left(\frac{d^2V}{dx^2}\right)\frac{\delta x^2}{2.4} + \left(\frac{d^4V}{dx^4}\right)\frac{\delta x^4}{2.3.4.4^2} + \left(\frac{d^6V}{dx^6}\right)\frac{\delta x^6}{2\,.\,.\,.\,6.4^3}\,\&c. \;.\;.\;.\;.\;.\; (20).$$

And as the formula (19) will apply generally to any other function, it will give, for the curvilinear area contained between V_0 and V_1,

$$\delta(\textstyle\int V dx) = V_{\frac{1}{2}}\delta x + \left(\frac{d^2V}{dx^2}\right)\frac{\delta x^3}{2.3.4} + \left(\frac{d^4V}{dx^4}\right)\frac{\delta x^5}{2\,.\,.\,5.4^2} + \left(\frac{d^6V}{dx^6}\right)\frac{\delta x^7}{2\,.\,.\,.\,7.4^3} \;\cdot\; (21).$$

Also the trapezoidal area between V_0 and V_1 is

$$(V)\delta x = V_{\frac{1}{2}}\delta x + \left(\frac{d^2V}{dx^2}\right)\frac{\delta x^3}{2.4} + \left(\frac{d^4V}{dx^4}\right)\frac{\delta x^5}{2.3.4.4^2} + \left(\frac{d^6V}{dx^6}\right)\frac{\delta x^7}{2\,.\,.\,.\,6.4^3}\,\&c.\; (22).$$

The curvilinear segment is found by subtracting (22) from (21), and is

$$\text{Segment} = -\left(\frac{d^2V}{dx^2}\right)\frac{\delta x^3}{3.4} - \left(\frac{d^4V}{dx^4}\right)\frac{\delta x^5}{2.3.5.4^2} - \left(\frac{d^6V}{dx^6}\right)\frac{\delta x^7}{2\,.\,.\,5.7.4^3}\,\&c.\,.\; (23).$$

But, according to the formula (19),

$$\begin{array}{llll}
\delta\left(\frac{dV}{dx}\right) = \left(\frac{d^2V}{dx^2}\right)\delta x & + \left(\frac{d^4V}{dx^4}\right)\frac{\delta x^3}{2.3.4} & + \left(\frac{d^6V}{dx^6}\right)\frac{\delta x^5}{2\,.\,.\,5.4^2} & +\&c. \\
\delta\left(\frac{d^3V}{dx^3}\right) = \;.\;.\;.\;.\;.\;. & \left(\frac{d^4V}{dx^4}\right)\delta x & + \left(\frac{d^6V}{dx^6}\right)\frac{\delta x^3}{2.3.4} & +\&c. \\
\delta\left(\frac{d^4V}{dx^4}\right) = \;.\;.\;.\;.\;.\;.\;.\;. & .\;.\;.\;.\;.\;.\;. & \left(\frac{d^6V}{dx^6}\right)\delta x & +\&c. \\
\&c. & \&c. & \&c. &
\end{array}$$

Therefore, by substitution in (23),

$$\text{Segment} = -\frac{\delta x^2}{12}\delta\left(\frac{dV}{dx}\right) + \frac{\delta x^4}{3.4.5.12}\delta\left(\frac{d^3V}{dx^3}\right) - \frac{\delta x^6}{3\ldots7.12}\delta\left(\frac{d^5V}{dx^5}\right)\text{\&c.} \;.\; (24).$$

and as the interval δx is constant, we hence obtain, by summation from V_0 to V_n,

$$\begin{aligned}\Sigma(\text{segments})_{0\ldots\ldots n} = & \frac{\delta x^2}{12}\left\{\left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_n\right\} \\ & -\frac{\delta x^4}{3.4.5.12}\left\{\left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_n\right\} \\ & +\frac{\delta x^6}{3\ldots7.12}\left\{\left(\frac{d^5V}{dx^5}\right)_0 - \left(\frac{d^5V}{dx^5}\right)_n\right\} \\ & \text{\&c.} \qquad \text{\&c.} \;.\;.\;.\;.\;.\; (25).\end{aligned}$$

If the interpolated interval $\frac{\delta x}{m}$ be substituted for δx, that is, if the respective terms of (25) be divided by m^2, m^4, m^6, &c., it will then express the value of the sum of the interpolated segments between the same limits. Now, as the total curvilinear area is made up either of the polygon $\Sigma(V)\delta x$ and the segments (25), or of the interpolated polygon $\Sigma(V')\frac{\delta x}{m}$ and the corresponding interpolated segments, it will follow that

$$\Sigma(V')\frac{\delta x}{m} = \Sigma(V)\delta x + \Sigma(\text{segments}) - \Sigma(\text{interpolated segments})$$

$$\begin{aligned}\therefore \frac{\Sigma(V')_{0\ldots\ldots n}}{m} = \Sigma(V) & + \frac{m^2-1}{12m^2}\delta x\left\{\left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_n\right\} \\ & -\frac{m^4-1}{720m^4}\delta x^3\left\{\left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_n\right\} \\ & +\frac{m^6-1}{30240m^6}\delta x^5\left\{\left(\frac{d^5V}{dx^5}\right)_0 - \left(\frac{d^5V}{dx^5}\right)_n\right\} \\ & \text{\&c.} \qquad \text{\&c.} \;.\;.\;.\;.\; (26).\end{aligned}$$

In the corresponding formula on page 317 the value of δx is supposed to be unity.

(*To be continued.*)